Dirichlet heat kernel estimates for unimodal Lévy processes

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Isotropic unimodal Lévy processes

Definition

A measure $m(dx)$ on $\mathbb{R}^d$ is called isotropic unimodal if $m(dx)$ is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial and radially non-increasing density.

A Lévy process X is called isotropic unimodal if a transition probability $p(t, dx) = \mathbb{P}(X_t \in dx)$ is isotropic unimodal, for all $t > 0$.

Remark

A pure jump Lévy process X is isotropic unimodal if and only if its Lévy measure $\nu(dx)$ is isotropic unimodal.

Let ψ be the Lévy exponent of pure jump isotropic unimodal Lévy process. Then,

$$\psi(\xi) = \int_{\mathbb{R}^d} \big(1 - \cos(\xi \cdot x)\big)\nu(|x|)dx, \tag{1}$$

where $\nu : (0, \infty) \to [0, \infty)$ is non-increasing function satisfying

$$\int_{\mathbb{R}^d} \min\{1, |x|^2\}\nu(|x|)dx < \infty. \tag{2}$$

Note that ψ is also radial function.

Conversely, for any non-increasing function ν satisfying (2), there is an isotropic unimodal Lévy process X with Lévy exponent ψ which is of the form (1).

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Subordinate Brownian motion (SBM)

- Let $B = (B_t,\ t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$, and $S = (S_t,\ t \geq 0)$ be a subordinator which is independent of B. The process $X = (X_t\ :\ t \geq 0)$ defined by
$$X_t = B_{S_t}$$
is a rotationally invariant Lévy process in $\mathbb{R}^d$ and is called a subordinate Brownian motion (SBM).

- Let ϕ be the Laplace exponent of S. That is,
$$\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}, \qquad \lambda > 0.$$
Then, it is known that ϕ is a Bernstein function. i.e., ϕ is smooth function that satisfies
$$(-1)^n \phi^{(n)} \leq 0, \quad \text{for all} \ \ n \geq 1.$$

- The Lévy exponent ψ is given by $\psi(\xi) = \phi(|\xi|^2)$.

- Every Bernstein function with $\phi(0+) = 0$ has a representation

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(\mathrm{d}t),$$

where $b \geq 0$, and $\mu(\mathrm{d}t)$ is a measure satisfying

$$\int_0^\infty \min\{1, t\}\, \mu(\mathrm{d}t) < \infty\,,$$

which is called the Lévy measure of ϕ.

- The transition density of X is defined by

$$p(t,x) = \int_0^\infty (4\pi s)^{-d/2} \exp\left(-\frac{|x|^2}{4s}\right) \mathbb{P}(S_t \in \mathrm{d}s).$$

- The Lévy measure $\nu(\mathrm{d}x)$ of X has a density $\nu(|x|)$, where

$$\nu(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(\mathrm{d}t).$$

Auxiliary functions

For two functions $f, g \geq 0$, $f \asymp g$ means there exists $c > 0$ such that $c^{-1} \leq f/g \leq c$.
For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$.

Let X be a pure jump isotropic unimodal Lévy process with Lévy measure $\nu(|x|)dx$.
For $r > 0$, we define

$$K(r) := r^{-2} \int_{|y| \leq r} |y|^2 \nu(y) \mathrm{d}y, \quad h(r) := r^{-2} \int_{\mathbb{R}^d} \left(r^2 \wedge |y|^2 \right) \nu(y) \mathrm{d}y.$$

Since ν is non-increasing, we have

$$K(r) \geq c\nu(r) r^{-2} \int_0^r s^{d+1} \mathrm{d}s = c(d+2)^{-1} r^d \nu(r).$$

Then,

$$\psi(r^{-1}) \asymp h(r), \quad r > 0.$$

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Weak scaling condition

Suppose g is a function from $(0,\infty)$ to $(0,\infty)$. Let $a \geq 0$, $\alpha \in [0,2]$, and $0 < c \leq 1 \leq C$.

- We say that g satisfies weak lower scaling condition near ∞ with index α if

$$\frac{g(\lambda r)}{g(r)} \geq c\lambda^{\alpha} \quad \text{for all} \quad \lambda \geq 1,\ r > a.$$

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- When g satisfies $U^a(\alpha, C)$ (resp. $L^a(\alpha, c)$) with $a = 0$, then we say that g satisfies the *global* weak upper scaling condition $U(\alpha, C)$ (resp. the *global* weak lower scaling condition $L(\alpha, c)$.)

Suppose that X is a pure jump isotropic unimodal Lévy process whose Lévy exponent ψ satisfies $L(\alpha_1, c)$ and $U(\alpha_2, C)$. Consider the following three cases:

(Case1) $0 < \alpha_1 \leq \alpha_2 < 2$;

Ex) $\psi(\xi) = |\xi|^\alpha \quad \text{for} \quad 0 < \alpha < 2,$

$\psi(\xi) = |\xi|^\alpha + |\xi|^\gamma \quad \text{for} \quad 0 < \alpha \leq \gamma < 2;$

(Case2) $\alpha_2 = 2$;

Ex) $\psi(\xi) = \dfrac{|\xi|^2}{\log(1+|\xi|^2)} - 1,$

$\psi(\xi) = \dfrac{|\xi|^2}{\log(1+|\xi|^\alpha)} \quad \text{for} \quad 0 < \alpha < 2;$

(Case3) $\alpha_1 = 0$.

Ex) $\psi(\xi) = (\log(1+|\xi|^\alpha))^\gamma \quad \text{for} \ \ \alpha \in (0, 2] \ \ \text{and} \ \ \gamma \in (0, 1],$

$\psi(\xi) = \log(1 + \log(1 + |\xi|^\alpha)) \quad \text{for} \ \ \alpha \in (0, 2].$

Dirichlet heat kernel

Let X be a discontinuous Markov process in $\mathbb{R}^d$ with the infinitesimal generator $\mathcal{L}$. We assume $\mathbb{P}_x(X_t \in \mathrm{d}y)$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$.

Let $D \subset \mathbb{R}^d$ be an open set and X^D be a subprocess of X killed upon leaving D and $p_D(t,x,y)$ be a transition density of X^D. Then $p_D(t,x,y)$ describes the distribution of X^D, i.e., $\mathbb{P}_x(X_t^D \in A) = \int_A p_D(t,x,y)\mathrm{d}y$.

An infinitesimal generator $\mathcal{L}|_D$ of X^D is the infinitesimal generator $\mathcal{L}$ with zero exterior condition. $p_D(t,x,y)$ is also called the Dirichlet heat kernel for $\mathcal{L}|_D$ since

$$u(t,x) := \int_{\mathbb{R}^d} p_D(t,x,y)f(y)\mathrm{d}y$$

is the solution to exterior Dirichlet problem:

$$\begin{cases} \mathcal{L}u = \partial_t u, \quad u(0,x) = f(x) \quad \text{on} \quad D, \\ u = 0 \quad \text{on} \quad D^c. \end{cases}$$

Note that if X is a symmetric pure jump Lévy process, then

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d} \left(u(x+y) - u(x) - \nabla u \cdot y 1_{\{|y|<1\}}\right)\nu(\mathrm{d}y), \quad u \in C_c^2.$$

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Dirichlet heat kernel estimates(DHKE)

For open set D, let $\delta_D(x) = \text{dist}(x, \partial D)$ and $\tau_D := \inf\{t > 0 : X_t \notin D\}$.

Chen, Kim and Song ('10)

Let $T > 0$ and X be an α-stable process for $\alpha \in (0,2)$ in $C^{1,1}$ open set D. Then, for $(t,x,y) \in (0,T] \times D \times D$,

$$p_D(t,x,y) \asymp \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

Bogdan, Grzywny and Ryznar ('10)

A Varopoulos type factorization estimate for symmetric stable process in κ-fat open set: for $(t,x,y) \in (0,T] \times D \times D$,

$$p_D(t,x,y) \asymp \mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)p(t,x,y).$$

Extensions

- Chen, Kim and Song ('10): DHKE for censored stable-like processes in $C^{1,1}$ open sets.
- Chen and Tokle ('11): DHKE for α- stable process in exterior $C^{1,1}$ open sets and half-space-like open sets.
- Chen, Kim and Song ('14): A Varopoulos type factorization estimate for a rotationally symmetric Lévy processes in κ-fat open sets and DHKE in $C^{1,1}$ open sets.
- Bogdan, Grzywny and Ryznar ('14): DHKE for the isotropic unimodal Lévy process whose Lévy exponent satisfies weak scaling condition in $C^{1,1}$ open sets.
- Kim and Kim ('14): DHKE for symmetric Markov processes dominated by stable-like processes in $C^{1,\rho}$ open sets.
- Grzywny, Kim and Kim ('15): DHKE for symmetric Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions in $C^{1,\rho}$ open sets.
- Chen, Kim and Song ('16): DHKE for SBM with Gaussian components in $C^{1,1}$ open sets.
- Chen and Kim ('16): DHKE for symmetric Lévy processes in half space.
- Kim and Mimica ('17): DHKE for SBM whose upper scaling index of Lévy exponent is not necessarily strictly below 2 when D is $C^{1,1}$ open set.

Slowly varying function and de Haan class

For given function $\ell : (0,\infty) \to (0,\infty)$, we say that ℓ is slowly varying function at ∞ (resp. at 0) if for all $\lambda > 0$

$$\lim_{\substack{x\to\infty \\ (\text{resp. } x\to 0)}} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

We denote $\mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$) by the class of slowly varying functions at ∞ (resp. 0).

For $\ell \in \mathcal{R}_0^\infty$(resp. $\mathcal{R}_0^0$), we denote Π_ℓ^∞ (resp. Π_ℓ^0) by the class of real-valued measurable function f on $[c,\infty)$ (resp. $(0,c)$) such that for all $\lambda > 0$

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Π_ℓ^∞ (resp. Π_ℓ^0) is called de Haan class at ∞ (resp. 0) determined by ℓ. Note that $|f| \in \mathcal{R}_0^\infty$ for $f \in \Pi_\ell^\infty$.

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Results in [Grzywny, Ryznar and Trojan('18)]

Let Y be a pure jump isotropic unimodal Lévy process on $\mathbb{R}^d$ with Lévy exponent φ and Lévy measure σ.

Proposition

There exist $C, c > 0$ such that for all $t > 0$ and $r > 0$

$$C^{-1} e^{-c^{-1} t \varphi(r^{-1})} \leq \mathbb{P}_x(\tau_{B(x,r)} > t) \leq C e^{-ct\varphi(r^{-1})}.$$

Proposition

There exist $C, c > 0$ such that for all $t > 0$ and $x \in \mathbb{R}^d$

$$p^Y(t,x) \geq Ct\sigma(x) e^{-ct\varphi(|x|^{-1})}.$$

Theorem

There exist $C > 0$ such that for all $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$p^Y(t,x) \leq Ct|x|^{-d} K^Y(|x|).$$

Heat kernel estimates(Case3)

Theorem [Grzywny, Ryznar and Trojan('18)]

Suppose $\psi \in \Pi_\ell^\infty$ for bounded $\ell \in \mathcal{R}_0^\infty$. Then there are $r_0, t_0 > 0$ such that for all $t \in (0, t_0)$ and $0 < |x| \le r_0$,

$$p(t,x) \asymp t|x|^{-d}\ell(|x|^{-1})e^{-t\psi(|x|^{-1})}.$$

Corollary [Grzywny, Ryznar and Trojan('18)]

Suppose $\psi \in \Pi_\ell^\infty$ for bounded $\ell \in \mathcal{R}_0^\infty$. Then, there is $t_* > 0$ such that for all $t \in (0, t_*)$,

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Assumptions

(A-1) $\lim_{r\to 0} \frac{\nu(r)}{r^d \ell(r^{-1})} = c$ for bounded $\ell \in \mathcal{R}_0^\infty$ and $c > 0$;

(A-2) The Lévy measure ν is infinite;

(A-3) $r \mapsto \nu(r)$ is differentiable and $r \mapsto -\nu'(r)/r$ is decreasing;

(A-4) There exists a constant $C_0 > 0$ such that

$$\nu(r) \le C_0 \nu(r+1) \quad \text{for all} \quad r \ge 1;$$

(A-5) There exist constants $R_\infty, M_0 > 0$ and $\alpha < 2$ such that

$$\frac{\nu(s)}{\nu(r)} \le M_0 \left(\frac{s}{r}\right)^{-d-\alpha} \quad \text{for every} \quad R_\infty < s < r < \infty.$$

Remark

- **(A-1)** is equivalent to the condition $\psi \in \Pi^{\infty}_{c'\ell}$ for $c' > 0$;
- Every subordinate Brownian motion satisfies **(A-3)**.
- **(A-1)**, **(A-2)** and **(A-4)** imply the Harnack inequality and boundary Harnack principle.
- **(A-1)**–**(A-4)** give the gradient estimates of harmonic functions.
- We will use **(A-5)** only if D is unbounded.
- **(A-5)** is equivalent to the condition $c^{-1}r^d\nu(r) \le K(r) \le cr^d\nu(r)$ for all $r > R_\infty$. Also, **(A-5)** implies $\nu(r) \le M_0 2^{d+\alpha}\nu(2r)$ for all $r > R_\infty$. Thus, **(A-5)** implies **(A-4)**.

Let $diag = \{(x,x) : x \in \mathbb{R}^d\}$. Then, we obtain the following:

Let $T > 0$. Suppose that X is an isotropic unimodal Lévy process on $\mathbb{R}^d$ with Lévy density ν satisfying **(A-1)**. Then there exist $C, b, \bar{b} > 0$ such that for every $(t,x,y) \in (0,T] \times (\mathbb{R}^d \times \mathbb{R}^d \setminus diag)$,

$$p(t,x,y) \geq C^{-1} t\nu(|x-y|) e^{-bt\psi(|x-y|^{-1})}$$

and

$$p(t,x,y) \leq Ct \frac{K(|x-y|)}{|x-y|^d} e^{-\bar{b}t\psi(|x-y|^{-1})}.$$

Thus, if $K(r) \asymp r^d \nu(r)$ for $r > 0$, we have for $(t,x,y) \in (0,T] \times (\mathbb{R}^d \times \mathbb{R}^d \setminus diag)$,

$$C^{-1} t\nu(|x-y|) e^{-bt\psi(|x-y|^{-1})} \leq p(t,x,y) \leq Ct\nu(|x-y|) e^{-\bar{b}t\psi(|x-y|^{-1})}.$$

Survival probability estimates

Proposition [Cho, K and Kim]

Suppose **(A-1)–(A-4)** holds. Let $T > 0$ and D be a $C^{1,\rho}$ open set in $\mathbb{R}^d$ for $\rho \in (0,1]$. Then,

$$\mathbb{P}_x(\tau_D > t) \asymp \left(1 \wedge \frac{\psi(\delta_D(x)^{-1})^{-1}}{t}\right)^{1/2},$$

for all $(t,x) \in (0,T] \times D$.

Idea of Proof. Since we consider $C^{1,\rho}$ open set with $\rho \in (0,1]$, we use the approach in Grzywny, Kim and Kim ('15). But, in Grzywny, Kim and Kim ('15), they assumed that weak lower scaling condition for the small jumps. Since we do not have such lower scaling condition in our case, their approach could not be applied directly. To overcome this difficulty, we use Harnack inequality and boundary Harnack principle to obtain survival probability estimates, which were not used in Grzywny, Kim and Kim ('15).

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Dirichlet heat kernel estimates (Case 3)

Theorem[Cho, K and Kim]

Suppose that X is an isotropic unimodal Lévy process with Lévy exponent ψ and Lévy measure ν satisfying conditions **(A-1)–(A-4)**. For $0 < \rho \le 1$, let D be a $C^{1,\rho}$ open subset of $\mathbb{R}^d$ with characteristics (R_1, Λ). If D is unbounded, we further assume that **(A-5)** holds. Then, for $T > 0$, there exist positive constants c_i $(i = 1, 2, 3, 4)$, which depend on $\rho, R_1, \Lambda, T, d, \ell, C_0, M_0$ such that the following estimate holds:

(i) For all $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t,x,y) \le c_1\left(1 \wedge \frac{\psi(\delta_D(x)^{-1})^{-1}}{t}\right)^{1/2}\left(1 \wedge \frac{\psi(\delta_D(y)^{-1})^{-1}}{t}\right)^{1/2} t\nu(|x-y|)e^{-c_2 t\psi(|x-y|^{-1})}.$$

(ii) For all $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t,x,y) \ge c_3\left(1 \wedge \frac{\psi(\delta_D(x)^{-1})^{-1}}{t}\right)^{1/2}\left(1 \wedge \frac{\psi(\delta_D(y)^{-1})^{-1}}{t}\right)^{1/2} t\nu(|x-y|)e^{-c_4 t\psi(|x-y|^{-1})}.$$

Difficulties in the proof. For the lower bound, if ψ satisfies the weak lower scaling condition, then the following argument can be applied:

$$\begin{aligned} p_D(t,x,y) &= \int_{D\times D} p_D(t/3,x,u)p_D(t/3,u,v)p_D(t/3,v,y)\mathrm{d}u\mathrm{d}v \\ &\geq \inf_{(u,v)\in B(A_x,\kappa r/6)\times B(A_y,\kappa r/6)} p_D(t/3,u,v) \\ &\quad \times \int_{B(A_x,\kappa r/6)} p_D(t/3,x,u)\mathrm{d}u \int_{B(A_y,\kappa r/6)} p_D(t/3,v,y)\mathrm{d}v, \end{aligned}$$

where $r = \frac{\psi^*(t^{-1})^{-1}}{\psi^*(T^{-1})^{-1}} R$ and for some $A_x = A_x(r), A_y = A_y(r) \in D$.

In our case, the above method cannot be applied directly. Thus, in our case, we use

$$p_D(t,x,y) \geq ct\nu(|x-y|)\mathbb{P}_x(\tau_U > t)\mathbb{P}_y(\tau_V > t),$$

for some proper open subset of $U, V \subset D$.

Example 1

Let Y be an isotropic α-stable process and S be a subordinator with the Laplace exponent $\phi(\lambda) = (\log(1+\lambda))^\gamma$ for $\gamma \in (0,1]$. Then $X = (X_t : t \geq 0)$ where $X_t = Y_{S_t}$ has the Lévy-Khintchine exponent

$$\psi(\xi) = (\log(1+|\xi|^\alpha))^\gamma, \quad \text{for } \alpha \in (0,2],\ \gamma \in (0,1],\ \alpha\gamma < 2.$$

Then, $\psi \in \Pi_\ell^\infty$ for

$$\ell(\lambda) = \alpha^\gamma \gamma (\log(1+\lambda^\alpha))^{\gamma-1}, \quad \lambda > 1.$$

Thus, for $T > 0$ and $C^{1,\rho}$ open set D, we have

$$\begin{aligned} p_D(t,x,y) \leq c_1 &\left(1 \wedge \frac{(\log(1+\delta_D(x)^{-\alpha}))^{-\gamma/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{(\log(1+\delta_D(y)^{-\alpha}))^{-\gamma/2}}{\sqrt{t}}\right) \\ &\times \frac{t}{|x-y|^d}(\log(1+|x-y|^{-\alpha}))^{\gamma-1} e^{-c_2 t(\log(1+|x-y|^{-\alpha}))^\gamma} \end{aligned}$$

and

$$\begin{aligned} p_D(t,x,y) \geq c_3 &\left(1 \wedge \frac{(\log(1+\delta_D(x)^{-\alpha}))^{-\gamma/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{(\log(1+\delta_D(y)^{-\alpha}))^{-\gamma/2}}{\sqrt{t}}\right) \\ &\times \frac{t}{|x-y|^d}(\log(1+|x-y|^{-\alpha}))^{\gamma-1} e^{-c_4 t(\log(1+|x-y|^{-\alpha}))^\gamma}. \end{aligned}$$

Example 2

Let Y be an isotropic α-stable process and S be an iterated geometric subordinator with the Laplace exponent $\phi(\lambda) = \log\big(1+\log(1+\lambda)\big)$. Then $X = (X_t : t \geq 0)$ where $X_t = Y_{S_t}$ has the Lévy-Khintchine exponent

$$\psi(x) = \log\big(1+\log(1+|x|^{\alpha})\big), \quad \text{for} \ \ \alpha \in (0,2).$$

Then, $\psi \in \Pi_{\ell}^{\infty}$ for

$$\ell(\lambda) = \big(1+\log(1+\lambda^{\alpha})\big)^{-1}, \quad \lambda \geq 1.$$

Thus, for $T > 0$ and $C^{1,\rho}$ open set D, there exist c_i $(i = 1,2,3,4)$ such that

$$\begin{aligned} &p_D(t,x,y) \\ &\leq c_1\left(1 \wedge \frac{(\log(1+\log(1+\delta_D(x)^{-\alpha})))^{-1/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{(\log(1+\log(1+\delta_D(y)^{-\alpha})))^{-1/2}}{\sqrt{t}}\right) \\ &\quad \times t|x|^{-d}\big(1+\log(1+|x-y|^{-1})\big)^{-1-c_2 t} \end{aligned}$$

and

$$\begin{aligned} &p_D(t,x,y) \\ &\geq c_3\left(1 \wedge \frac{(\log(1+\log(1+\delta_D(x)^{-\alpha})))^{-1/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{(\log(1+\log(1+\delta_D(y)^{-\alpha})))^{-1/2}}{\sqrt{t}}\right) \\ &\quad \times t|x|^{-d}\big(1+\log(1+|x-y|^{-1})\big)^{-1-c_4 t}. \end{aligned}$$

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Thank you